Announcements

- Prelim 2 results will open after class: mean 54.38 (out of 70), stdev 11.22)
- Next plans: Spring break
- Monday/Tuesday April 6-7 section on divide and conquer,

  no quiz
- HW8 is posted: divide and conquer, due Friday April 10

today's OH: 10-11 Gates 316

Convolution and polynomial multiplication: recall plan

given $f(x) = a_0 + a_1 x + \dots a_{n-1} x^{n-1}$

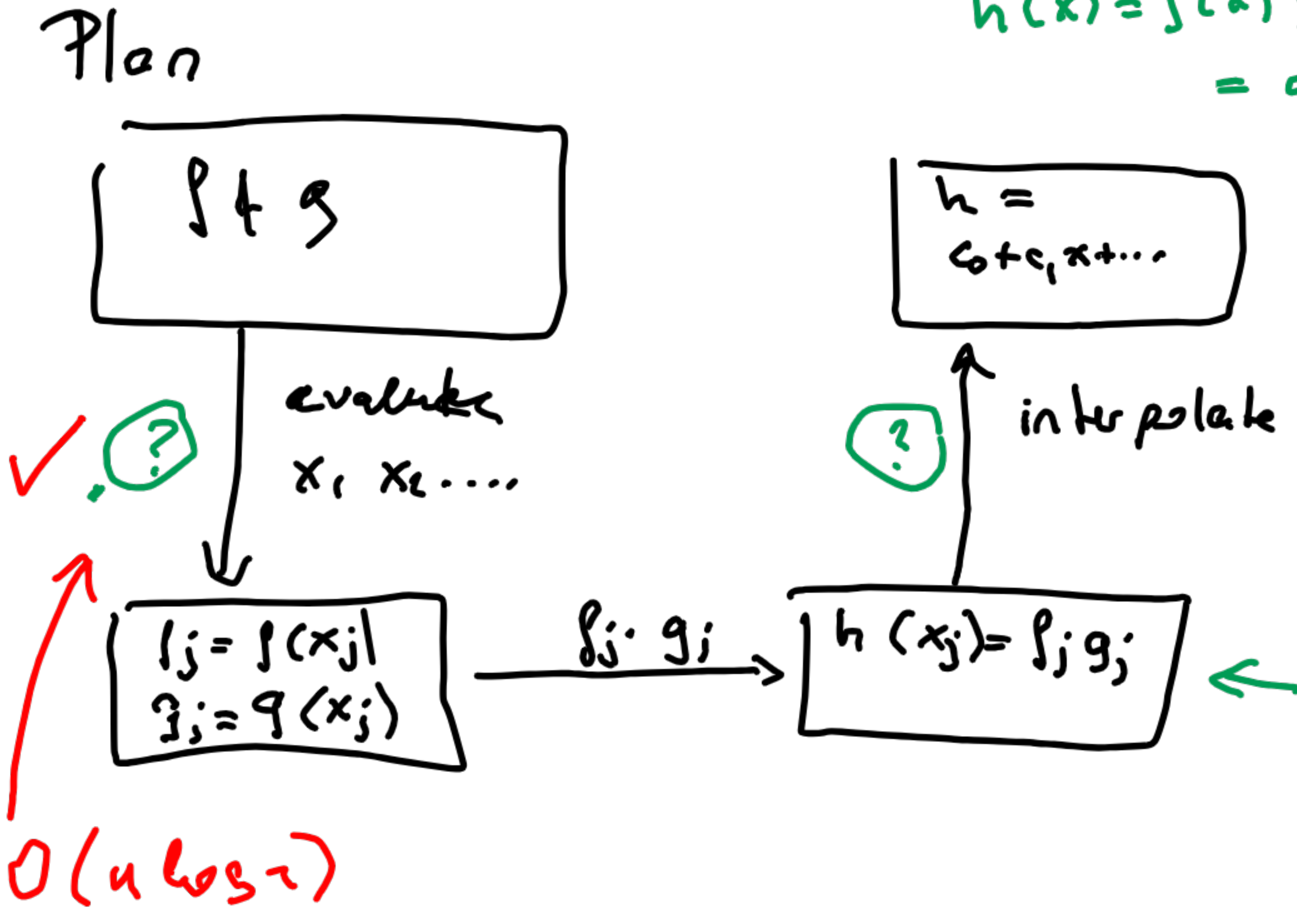
$g(x) = b_0 + b_1 x + \dots b_{n-1} x^{n-1}$

n coefficients

$h(x) = f(x)\, g(x)$

$= a_0 b_0 + (a_1 b_0 + b_1 a_0)\, x + \dots \Big(\underbrace{\sum_{\substack{j,\ell \\ j+\ell=k}} a_j b_\ell}_{\text{convolution}}\Big) x^k + \dots$

convolution = compute these sums for all k



degree $2n-2$

$2n-1$ coefficients $\Rightarrow$ need $\geq 2n-1$ x values

Divide and Conquer for evaluating a polynomial

$f(x) = a_0 + a_1 x + \dots + a_{n-1} x^{n-1}$ evaluate at x

$f(x) = (a_0 + a_2 x^2 + a_4 x^4 + \dots) + x(a_1 + a_3 x^2 + a_5 x^4 + \dots)$

$f(x) = f_e(x^2) + x\, f_o(x^2)$ f_e & f_o degree $\le \frac{n-1}{2}$, $\le \frac{n}{2}$ coefficient

D&C algorithm evaluate f_e & f_o on x^2 &

time $T(n) = 2T(\frac{n}{2}) + O(1)$

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Running time for evaluating a degree n polynomial (assume n is a power of 2)

A. $O(1)$

B. $O(\log n)$

C. $O(n)$

D. $O(n \log n)$

E. I don't know

$T(n) = 2T\left(\frac{n}{2}\right) + c$ constant c

$T(n) = q\,T(n/2) + c \cdot n^{\alpha}$ $q = 2$, $\alpha = 0$

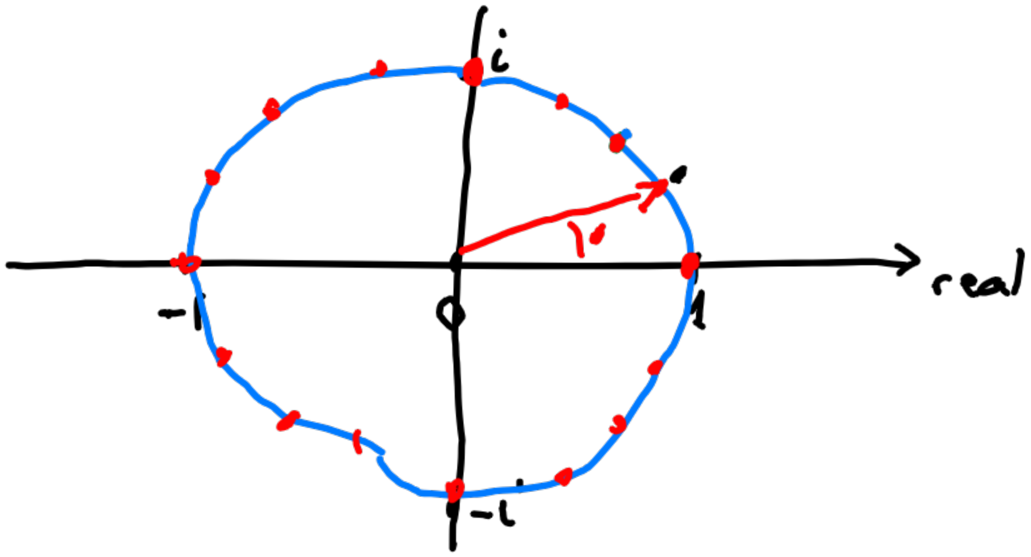
$O(n^{\log_2 q}) = O(n)$ $2^{\alpha} < q$

Polynomial interpolation for values that are the roots of unity

complex root of 1

$x^n = 1$ solutions $1, -1, i, -i$

$(-i)^2 = i^2 = -1$ $(-i)^4 = 1$



$e^{2\pi i/n \cdot l}$ nth root of unit (integer

angle in polar coordinate representation

$$\left(e^{2\pi i/n\, l}\right)^n = e^{2\pi i l} = \left(e^{2\pi i}\right)^l = 1^l = 1$$

x values to use roots of unity with n power of 2 squares of roots of unity

$$\left(e^{2\pi i/n \cdot \ell}\right)^2 = e^{\frac{2\pi i \ell}{n/2}} = n/2 \text{ root of unity} \quad n/2 \text{ of them}$$

$$f(x_\ell) = \underbrace{f_e(x_\ell^2)}_{\text{degree } n/2 \text{ \& only } n/2 \text{ values to compute}} + x_\ell \cdot f_o(x_\ell^2) \quad \text{all } \ell$$

$T(n)$ = time to compute all n values (n power of 2)

$$T(n) = 2T(n/2) + O(n)$$

2 multiply + 1 add for each

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Running time for evaluating a degree n polynomial at the n^{th} roots of unity (assume n is a power of 2)

$T(n) = 2T(n/2) + c \cdot n$ c constant

$T(n) = a T(n/2) + c\, n^{\alpha}$ $a=2$, $\alpha=1$

$2^{\alpha} = a$

$T(n) = O(n \log n)$

A. $O(n)$

B. $O(n \log n)$

C. $O(n^2)$

D. $O(n^2 \log n)$

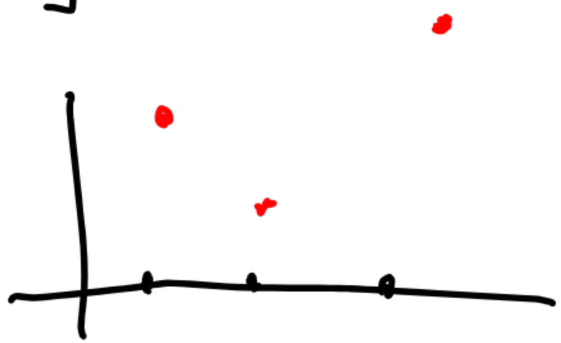
E. I don't know

Finding the polynomial given its values at the roots of unity

Polynomial $h(x) = c_0 + c_1 x + \dots + c_{m-1} x^{m-1}$

given $h(x_\ell) = h_\ell$ on x_ℓ root of unity

goal: recover c_j values



$D(x) = \sum_\ell h_\ell x^\ell$

Majic fact $D(x_k) = c_k$

$x_k = k^{th}$ root of unity $\Longrightarrow O(n \log n)$ time

x is n^{th} root of unity $\quad x^n - 1 = 0$

$$\Longrightarrow x^n - 1 = (x-1) \sum_{k=0}^{n-1} x^k$$

$$\boxed{\text{if } x \neq 1 \Longrightarrow \sum_{k=0}^{n-1} x^k = 0}$$

Consider $\boxed{h(x) = c_e x^e}$ only term to simplify

we have the $\underline{c_e} x^e$ when x is $\underline{k}$th root of unity : $h(x_s) = h_s$

$$D(x_k) = \sum_s h_s x_k^s = \sum_s h(x_s) \cdot x_k^s = \sum_s c_e x_s^e \cdot x_k^s$$

x_s & x_k are s & k root of unity

$$= c_e \sum_s x_s^e x_k^s$$

Claim $$\sum_s x_s^e x_k^s = \begin{cases} 0 & \text{if } k+e = n \\ 1 & k = 0 \end{cases}$$

$$x_s = x_1^s$$

$$\sum_s x_s^e x_k^s = \sum_s x_1^{se} x_1^{ks} = \sum_s x_1^{s(e+k)} = \sum_s (x_1^{e+k})^s = \sum_s x_{e+k}^s$$

0 unless $x_{e+k} = 1$